Lecture 9

13.3 - Arc Length and Curvature

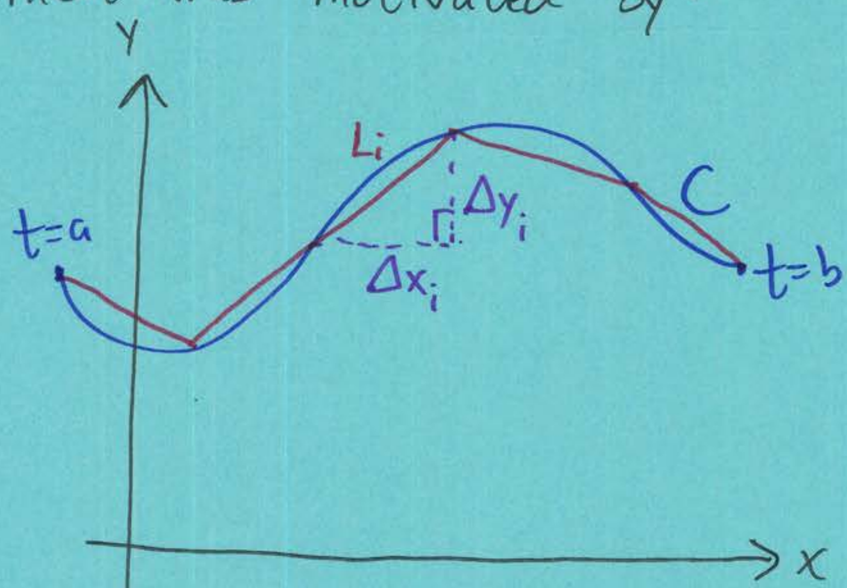
In Calculus II, you found the arc length of a plane curve $\vec{r}(t) = \langle f(t), g(t) \rangle$, $a \le t \le b$, as

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

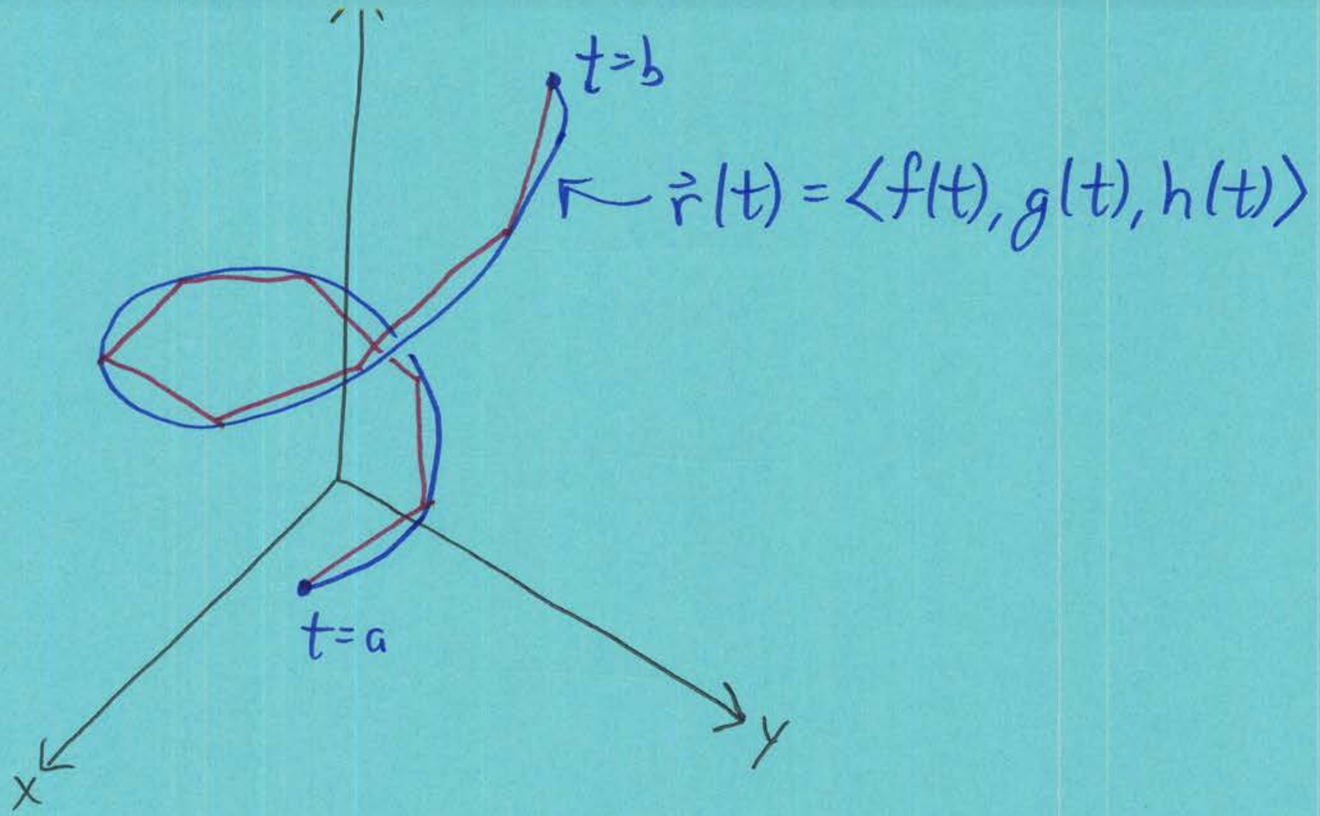
$$= \int_a^b \sqrt{dx^2 + dy^2}$$

which was motivated by:



$$L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

So, $L \approx \sum_i L_i = \sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$



We end up with $L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$

So, taking finer partitions and a limit:

$$L = \int_a^b \sqrt{dx^2 + dy^2 + dz^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

is the <u>arc length</u> of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ from $t=a$ to $t=b$

Some technical assumptions in that formula:

- $\vec{r}(t)$ cannot cross over itself for any $a \le t \le b$
- f', g', and h' must be continuous.

Re-examining the formula for arclength, since

$$|\vec{r}\,'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

we have

$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\, dt}$$

Ex: Find the arc length of

$$\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle, \quad -5 \le t \le 5$$

Sol: $\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t \rangle$

$$|\vec{r}\,'(t)| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$$

So $L = \int_{-5}^{5} \sqrt{10}\, dt = 10\sqrt{10}$. ◇

A curve C need not only be represented by one function, in fact they are represented by many. E.g.,

$$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad 1 \le t \le 2$$

is also represented by

$$\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle, \quad 0 \le u \le \ln 2.$$

$\vec{r}_1$ & $\vec{r}_2$ are called <u>parametrizations</u> of the curve C. One particular parametrization of interest is the following: Suppose C is given by

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, \quad a \le t \le b.$$

with $\vec{r}\,'$ continuous and $\vec{r}$ traverses C exactly once.

Define the <u>arc length function</u>

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\,du \qquad (*)}$$

This gives the length of the curve from $\vec{r}(a)$ to $\vec{r}(t)$.

By the fundamental theorem of calculus,

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

Suppose we can solve $(*)$ for t in terms of s: $t = t(s)$

We call the reparametrization of $\vec{r}$ in terms of s:

$$\vec{r} = \vec{r}(t(s))$$

the <u>parametrization with respect to arc length</u>.

<u>Ex</u>: Reparametrize $\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle$, $-5 \le t \le 5$ with respect to arc length.

<u>Sol</u>: From the last example,

$$s(t) = \int_{-5}^{t} \sqrt{10}\, du = \sqrt{10}\, t + 5\sqrt{10}$$

$$\Rightarrow \quad t = \frac{s - 5\sqrt{10}}{\sqrt{10}}$$

We need new bounds for s:

$$t = -5 \Rightarrow s = 0 \ , \quad t = 5 \Rightarrow s = 10\sqrt{10}$$

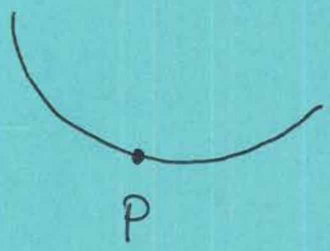
So, the reparametrization is:

$$\vec{r}(t(s)) = \left\langle \frac{s-5\sqrt{10}}{\sqrt{10}},\ 3\cos\left(\frac{s-5\sqrt{10}}{\sqrt{10}}\right),\ 3\sin\left(\frac{s-5\sqrt{10}}{\sqrt{10}}\right) \right\rangle$$

◇

Curvature

Intuitively, curvature is a measure of how sharply a curve bends, e.g.,



has a smaller curvature at P

than

does.

Technical def: A parametrization $\vec{r}(t)$ is called smooth on an interval I if $\vec{r}\,'$ is continuous and $\vec{r}\,'(t) \neq \vec{0}$ $\forall t \in I$. A curve C is called smooth if it has a smooth parametrization.

We define the curvature, $\kappa(t)$, to be the rate of change of the unit tangent vector w.r.t. arc length, that is: the curvature is:

$$\boxed{\kappa = \left| \frac{d\vec{T}}{ds} \right|}$$

$$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} = \frac{dT}{ds} |\vec{r}\,'(t)|$$

So, a more convenient formula for curvature is

$$\boxed{\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}}$$

<u>Ex</u>: Show that the curvature of a circle of radius a is $\frac{1}{a}$.

<u>Sol</u>: $\vec{r}(t) = \langle a\cos t, a \sin t \rangle$

$\vec{r}\,'(t) = \langle -a\sin t, a\cos t \rangle$, $|\vec{r}\,'(t)| = a$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin t, \cos t \rangle$,

$\vec{T}'(t) = \langle -\cos t, -\sin t \rangle$, $|\vec{T}'(t)| = 1$

So $\kappa(t) = \frac{1}{a}$.

$\diamond$

We can actually remove mention of $\vec{T}$ from κ:

Another formula for curvature

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

Ex: Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $(0,0,0)$.

Sol: We can do this in two ways:

i) Find a general formula for $\kappa(t)$, then plug in the appropriate t value.

ii) Find the relevant values at the t value, then compute κ at that t value.

We'll do (ii):

The t-value is $t=0$ $\left(\vec{r}(0) = \langle 0,0,0 \rangle\right)$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$, $\vec{r}\,'(0) = \langle 1,0,0 \rangle$, $|\vec{r}\,'(0)| = 1$

$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$, $\vec{r}\,''(0) = \langle 0,2,0 \rangle$

$\vec{r}\,'(0) \times \vec{r}\,''(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$, $|\vec{r}\,'(0) \times \vec{r}\,''(0)| = 2$

So, $\kappa(0) = \dfrac{|\vec{r}'(0) \times \vec{r}''(0)|}{|\vec{r}'(0)|^3} = 2$

In the special case of a plane curve, $y = f(x)$, we have: parametrize it as: $\vec{r}(x) = x\hat{i} + f(x)\hat{j}$

then: $\vec{r}'(x) = \hat{i} + f'(x)\hat{j}$, $\vec{r}''(x) = f''(x)\hat{j}$, and

$$\vec{r}'(x) \times \vec{r}''(x) = f''(x)\hat{k}.$$

So

$$\kappa(x) = \frac{|\vec{r}'(x) \times \vec{r}''(x)|}{|\vec{r}'(x)|^3} = \frac{|f''(x)|}{\sqrt{1+[f'(x)]^2}} = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}}$$

Frenet-Serret Frame (T-N-B frame)

We've already encountered one of these: $\vec{T}$.

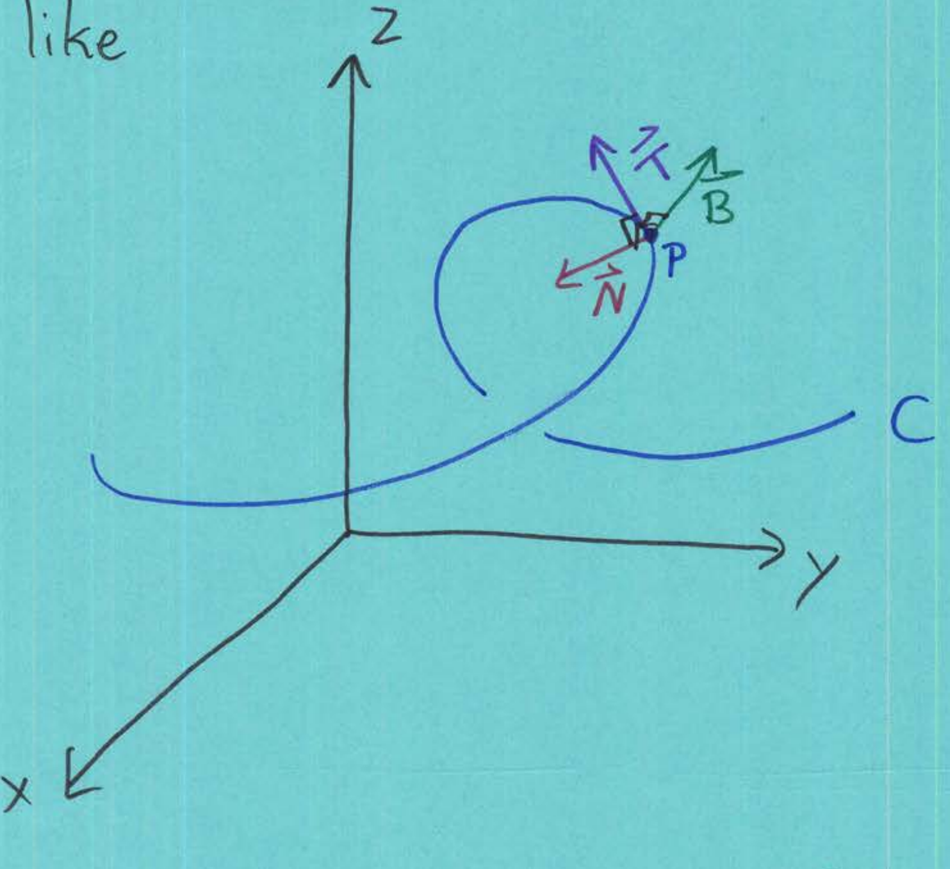
Since $|\vec{T}(t)| = 1 \Rightarrow \vec{T}(t) \perp \vec{T}'(t)$.

We define the <u>unit normal vector</u>: $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ (requires $\kappa(t) \neq 0$)

and the <u>binormal vector</u>: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ (also a unit vector!)

Sometimes, with this definition, $\vec{N}$ & $\vec{B}$ can be hard to compute. Another way to compute them is: $\boxed{\vec{B}(t) = \dfrac{\vec{r}'(t) \times \vec{r}''(t)}{|\vec{r}'(t) \times \vec{r}''(t)|}, \quad \vec{N}(t) = \vec{B}(t) \times \vec{T}(t)}$

At a point P along a curve C, these vectors look like



$\vec{N}$ always points in the direction the curve is bending.

These vectors also determine planes:

- The plane through P determined by $\vec{N}$ & $\vec{B}$ (meaning it contains $\vec{N}$ & $\vec{B}$, so $\vec{T}$ is a vector perpendicular to it) is called the <u>normal plane</u> of C at P. It contains all lines perpendicular to C at P.
- The plane determined by $\vec{T}$ & $\vec{N}$ (has normal vector $\vec{B}$) is called the <u>osculating plane</u> of C at P. It is the plane that comes closest to containing the curve at P.

In the osculating plane, there is a circle passing through P, which lies on the concave side of C (same side as $\vec{N}$ points towards) and has radius $\rho = \frac{1}{\kappa}$. This is the <u>osculating circle</u> and it is the circle which best describes how C behaves near P, i.e., it has the same tangent & normal vectors and the same curvature as C at P.

<u>Ex</u>: Find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ for $\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle$ and give equations for the normal and osculating plane at $(\frac{\pi}{2}, 0, 3)$.

<u>Sol</u>: $\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t \rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{10}$

$\vec{r}\,''(t) = \langle 0, -3\cos t, -3\sin t \rangle$, $\cancel{\Rightarrow |\vec{r}\,''(t)|}$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3\sin t & 3\cos t \\ 0 & -3\cos t & -3\sin t \end{vmatrix} = \langle 9\sin^2 t + 9\cos^2 t, 3\sin t, -3\cos t \rangle$$

$$= \langle 9, 3\sin t, -3\cos t \rangle$$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{81 + 9} = 3\sqrt{10}$$

Thus,

$$\underline{\vec{T}(t)} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left\langle \frac{1}{\sqrt{10}}, -\frac{3\sin t}{\sqrt{10}}, \frac{3\cos t}{\sqrt{10}} \right\rangle = \underline{\frac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle}$$

$$\underline{\vec{B}(t)} = \frac{\vec{r}'(t)\times\vec{r}''(t)}{|\vec{r}'(t)\times\vec{r}''(t)|} = \left\langle \frac{3}{\sqrt{10}}, \frac{\sin t}{\sqrt{10}}, -\frac{\cos t}{\sqrt{10}} \right\rangle = \underline{\frac{1}{\sqrt{10}}\langle 3, \sin t, -\cos t\rangle}$$

$$\underline{\vec{N}(t)} = \vec{B}(t)\times\vec{T}(t) = \left(\frac{1}{\sqrt{10}}\right)^2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & \sin t & -\cos t \\ 1 & -3\sin t & 3\cos t \end{vmatrix}$$

$$= \frac{1}{10}\langle 3\sin t\cos t - 3\cos t\sin t, -10\cos t, -10\sin t\rangle$$

$$= \underline{\langle 0, -\cos t, -\sin t\rangle}$$

At $(\frac{\pi}{2}, 0, 3)$, $t = \frac{\pi}{2}$, so: $\vec{T}(\frac{\pi}{2}) = \langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\rangle$ &

$\vec{N}(\frac{\pi}{2}) = \langle 0, 0, -1\rangle$ &

$\vec{B}(\frac{\pi}{2}) = \langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0\rangle$

To find the planes, we only need vectors $/\!/ \vec{T}$ & $\vec{B}$.

Normal plane: can use $\vec{v}_T = \langle 1, -3, 0\rangle$ $/\!/ \vec{T}(\frac{\pi}{2})$.

eqn is: $1(x - \frac{\pi}{2}) - 3(y - 0) + 0(z - 3) = 0$

$$\Leftrightarrow$$

$$\boxed{x - 3y - \frac{\pi}{2} = 0}$$

Osculating plane: can use $\vec{v}_B = \langle 3, 1, 0 \rangle \parallel \vec{B}$

eqn is: $3\left(x - \frac{\pi}{2}\right) + 1(y - 0) + 0(z - 3) = 0$

$$\Longleftrightarrow$$

$$\boxed{3x + y - \frac{3\pi}{2} = 0}$$

$\diamond$